TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 154, February 1971

CONDITIONS ON AN OPERATOR IMPLYING $\operatorname{Re}\sigma(T)=\sigma(\operatorname{Re} T)$

BY
S. K. BERBERIAN

Abstract. It is shown that the equation of the title is valid for certain classes of not necessarily normal operators (including Toeplitz operators, and operators whose spectrum is a spectral set), and a new proof is given of C. R. Putnam's theorem that it is valid for seminormal operators.

If T is any operator (bounded linear, in Hilbert space) then $f(\sigma(T))=\sigma(f(T))$ for every rational function $f(\lambda)$ without poles in $\sigma(T)$. On the other hand, if T is normal then for every polynomial $f(\lambda, \lambda^*)$ one has $\sigma(f(T))=f(\sigma(T))=\{f(\lambda, \lambda^*) : \lambda \in \sigma(T)\}$ (see [16] for a recent elegant proof). In the present paper, we are concerned with the polynomial $f(\lambda, \lambda^*)=\frac{1}{2}(\lambda+\lambda^*)=\operatorname{Re}\lambda$ for not necessarily normal operators T, i.e., we consider conditions on an operator T such that

$$(*) \qquad \operatorname{Re}\sigma(T)=\sigma(\operatorname{Re} T).$$

This is a very humble "spectral mapping theorem", but it is a considerable achievement for a nonnormal operator. For example, the fact that $(*)$ holds for T seminormal, which is due to C. R. Putnam [10], plays a role in his recent proof that a seminormal operator whose spectrum has zero area is normal [13]. In Theorem 1 we give a simple new proof of Putnam's theorem, and condition $(*)$ is verified for some classes of not necessarily seminormal operators in Theorems 2–4. Specifically, we prove

THEOREM 1 (PUTNAM). *If T is a seminormal operator, then $(*)$ holds.*

THEOREM 2. *If T is a Toeplitz operator, then $(*)$ holds.*

THEOREM 3. *If T is an operator such that $\sigma(T)$ is a spectral set for T, then $(*)$ holds.*

THEOREM 4. *If T satisfies the growth condition (G_1) and $\sigma(T)$ is connected, then $(*)$ holds.*

All operators contemplated in Theorems 1–4 are convexoid, i.e., $\operatorname{conv}\sigma(T)=\operatorname{Cl} W(T)$. (Here conv denotes convex hull, Cl denotes closure, and $W(T)=\{(Tx|x) : \|x\|=1\}$ is the numerical range of T.) Lemmas 4–6 give results weaker than $(*)$ for certain convexoid operators.

Received by the editors March 18, 1970.

AMS 1969 *subject classifications.* Primary 4730, 4740.

Key words and phrases. Hilbert space, spectrum, spectral mapping theorem, seminormal operator, Toeplitz operator, numerical range.

LEMMA 1. *If T is hyponormal then* $\operatorname{Re}\pi(T)\subset\sigma(\operatorname{Re}T)$.

Proof. Let $\lambda=\alpha+i\beta\in\pi(T)$, the approximate point spectrum of T, and let x_n be a sequence of unit vectors such that $Tx_n-\lambda x_n\to 0$. By hyponormality, $\|(T-\lambda I)^*x_n\|\leqq\|(T-\lambda I)x_n\|$, therefore also $T^*x_n-\lambda^*x_n\to 0$. Then $(\operatorname{Re}T)x_n-\alpha x_n =\frac{1}{2}(T+T^*)x_n-\frac{1}{2}(\lambda+\lambda^*)x_n\to 0$, thus $\alpha\in\sigma(\operatorname{Re}T)$. ∎

LEMMA 2. *If T is seminormal then* $\operatorname{Re}\sigma(T)\subset\sigma(\operatorname{Re}T)$.

Proof. One can suppose T hyponormal. Let $\lambda_0\in\sigma(T)$. The vertical line $\operatorname{Re}\lambda =\operatorname{Re}\lambda_0$ exits the spectrum at a boundary point μ of $\sigma(T)$. Since $\partial\sigma(T)\subset\pi(T)$, one has $\operatorname{Re}\lambda_0=\operatorname{Re}\mu\in\sigma(\operatorname{Re}T)$ by Lemma 1. ∎

LEMMA 3. *If T is normal then* $(*)$ *holds.*

Proof. This is immediate from the continuous functional calculus; see also the remarks at the beginning of the paper. Alternatively, see Theorem 3. ∎

Proof of Theorem 1. One can suppose T hyponormal. In view of Lemma 2, it suffices to show that $\sigma(\operatorname{Re}T)\subset\operatorname{Re}\sigma(T)$. Write $T=H+iJ$, H and J selfadjoint, and let $D=T^*T-TT^*$; $D\geqq 0$ by hypothesis, and one has

$$HJ-JH = -\tfrac{1}{2}iD. \tag{1}$$

Changing Hilbert space [2], one can suppose $\sigma(H)=\pi_0(H)$ (the point spectrum of H).

Assuming $\alpha\in\sigma(H)$, it is to be shown that $\alpha\in\operatorname{Re}\sigma(T)$. Let $\mathscr{M}=N(H-\alpha I)$, the null space of $H-\alpha I$; since α is an eigenvalue, $\mathscr{M}\neq\{0\}$. We show that $\mathscr{M}$ is invariant under J. Let $x\in\mathscr{M}$, i.e., $(H-\alpha I)x=0$. By (1),

$$(H-\alpha I)J-J(H-\alpha I) = -\tfrac{1}{2}iD,$$

therefore

$$-\tfrac{1}{2}i(Dx|x) = (Jx|(H-\alpha I)x)-(J(H-\alpha I)x|x) = 0,$$

thus $0=(Dx|x)=\|D^{1/2}x\|^2$, $D^{1/2}x=0$, $Dx=0$. Then (1) yields

$$0 = H(Jx)-J(Hx) = H(Jx)-\alpha(Jx),$$

thus $Jx\in\mathscr{M}$.

Let $J_1=J|\mathscr{M}$. Of course $\mathscr{M}$ is also invariant under H, with $H|\mathscr{M}=\alpha I$. Therefore $\mathscr{M}$ is invariant under $T=H+iJ$, and $T|\mathscr{M}=\alpha I+iJ_1$; since J_1 is selfadjoint, clearly $T|\mathscr{M}$ is normal; since T is hyponormal, it follows that $\mathscr{M}$ reduces T [1, p. 160, Example 9]. Writing $T_1=T|\mathscr{M}$ and $T_2=T|\mathscr{M}^\perp$, we have $T=T_1\oplus T_2$, $\sigma(T) =\sigma(T_1)\cup\sigma(T_2)$, therefore

$$\operatorname{Re}\sigma(T) = \operatorname{Re}\sigma(T_1)\cup\operatorname{Re}\sigma(T_2). \tag{2}$$

Since T_1 is normal, by Lemma 3 we have

$$\operatorname{Re}\sigma(T_1) = \sigma(\operatorname{Re}T_1) = \sigma(\alpha I) = \{\alpha\},$$

therefore $\alpha\in\operatorname{Re}\sigma(T)$ by (2). ∎

REMARK. The proof of Theorem 1 (and of Lemma 8 below) uses the fact that the representation $T\to T^0$ of [2] preserves spectrum; this fact is not proved explicitly in [2], but it follows at once from the observation that T is invertible iff T^*T and TT^* are invertible iff $T^*T\geqq cI$ and $TT^*\geqq cI$ for some $c>0$. {The underlying general principle is that if $b\in B\subset A$, where A is a C^*-algebra with unity and B is a closed $*$-subalgebra containing the unity, then $\sigma_B(b)=\sigma_A(b)$ [6, p. 8].}

LEMMA 4. *Let T be any convexoid operator. If* $[\alpha_0,\beta_0]$ *is the smallest interval containing* Re $\sigma(T)$, *then* $\alpha_0,\beta_0\in\sigma(\text{Re } T)$.

Proof. Changing Hilbert space, one can suppose that $W(T)$ is closed and that $\pi(T)=\pi_0(T)$ [5]. Choose $\lambda_0\in\sigma(T)$ with Re $\lambda_0=\alpha_0$. Clearly λ_0 is a boundary point of $\sigma(T)$ (by the extremality of α_0), therefore $\lambda_0\in\partial\sigma(T)\subset\pi(T)=\pi_0(T)$, hence also $\lambda_0\in W(T)$. On the other hand, since T is convexoid, $W(T)$ is closed and Re $\sigma(T)\geqq\alpha_0$, we have

$$\text{Re } W(T) = \text{Re conv } \sigma(T) \geqq \alpha_0;$$

thus λ_0 is clearly a boundary point of $W(T)$, therefore it is a normal eigenvalue of T, i.e., $N(T-\lambda_0 I)=N(T^*-\lambda_0^* I)$ [8, Satz 2]. Choose $u\neq 0$ with $Tu=\lambda_0 u$; then also $T^*u=\lambda_0^*u$, therefore $(\text{Re } T)u=\alpha_0 u$, thus $\alpha_0\in\sigma(\text{Re } T)$. Similarly $\beta_0\in\sigma(\text{Re } T)$. ∎

LEMMA 5. *Suppose T is convexoid and $\sigma(T)$ is connected. If* $[\alpha,\beta]$ *is the smallest interval containing* $\sigma(\text{Re } T)$, *then* $\sigma(\text{Re } T)\subset[\alpha,\beta]\subset\text{Re } \sigma(T)$.

Proof. Since $\sigma(T)$ is connected, Re $\sigma(T)$ is an interval, thus it will suffice to show that $\alpha,\beta\in\text{Re } \sigma(T)$. Assume to the contrary, e.g., that $\alpha\notin\text{Re } \sigma(T)$. Thus, if L is the vertical line Re $\lambda=\alpha$, then L is disjoint from $\sigma(T)$. Since $\sigma(T)$ is connected, it must lie strictly to one side of L. Suppose, e.g., that it lies to the right. Then there exists $\varepsilon>0$ such that Re $\sigma(T)\geqq\alpha+\varepsilon$. Since T is convexoid, it follows that Re $W(T)\geqq\alpha+\varepsilon$ thus $W(\text{Re } T)\geqq\alpha+\varepsilon$, i.e., Re $T\geqq(\alpha+\varepsilon)I$, hence $\sigma(\text{Re } T)\geqq\alpha+\varepsilon$; in particular, $\alpha\geqq\alpha+\varepsilon$, contrary to $\varepsilon>0$. The proof with "left" in place of "right" is similar, with $\alpha+\varepsilon$ replaced by $\alpha-\varepsilon$ and the inequalities reversed. ∎

LEMMA 6. *If T is a convexoid operator such that both $\sigma(T)$ and* $\sigma(\text{Re } T)$ *are connected, then* $(*)$ *holds.*

Proof. Say $\sigma(\text{Re } T)=[\alpha,\beta]$ and Re $\sigma(T)=[\alpha_0,\beta_0]$. By Lemma 5, $[\alpha,\beta]\subset[a_0,\beta_0]$; by Lemma 4, $\alpha_0,\beta_0\in[\alpha,\beta]$ and so $[\alpha_0,\beta_0]\subset[\alpha,\beta]$. ∎

Proof of Theorem 2. If T is a Toeplitz operator, then T is convexoid [7, Problem 196, Corollary 4] and $\sigma(T)$ is connected by a theorem of H. Widom [17]; since Re T is also a Toeplitz operator, $\sigma(\text{Re } T)$ is connected too (cf. [7, Problem 199]), thus Lemma 6 is applicable. ∎

LEMMA 7. *If T is a spectral set for T, then* $\sigma(\text{Re } T)\subset\text{Re } \sigma(T)$.

Proof. By hypothesis, $f(T)$ is normaloid for all rational functions f without poles in $\sigma(T)$ [3]. In particular, $T-\lambda I$ is normaloid for all complex λ, therefore T is convexoid [14], [8, Satz 9, (i)]. {Better yet, $(T-\lambda I)^{-1}$ is normaloid for all $\lambda \notin \sigma(T)$, i.e., T satisfies condition (G_1).}

Let $\alpha \in \sigma(\mathrm{Re}\, T)$ and assume to the contrary that $\alpha \notin \mathrm{Re}\, \sigma(T)$. Then, if L is the vertical line $\mathrm{Re}\, \lambda = \alpha$, L is disjoint from $\sigma(T)$.

Suppose first that $\sigma(T)$ lies to one side—say the left—of L. Then there exists $\varepsilon > 0$ such that $\mathrm{Re}\, \sigma(T) \leqq \alpha - \varepsilon$; since T is convexoid, the argument in Lemma 5 yields the contradiction $\alpha \leqq \alpha - \varepsilon$. Similarly if $\sigma(T)$ lies to the right of L.

Suppose finally that $\sigma(T)$ is disconnected by L. Write $\sigma(T) = X_1 \cup X_2$, where X_1 lies to the left, and X_2 to the right, of L. Since $\sigma(T)$ is a spectral set for T, by a theorem of J. P. Williams [18, Theorem 4] there is an orthogonal decomposition $T = T_1 \oplus T_2$ with $\sigma(T_k) = X_k$ a spectral set for T_k. Then $\mathrm{Re}\, T = \mathrm{Re}\, T_1 \oplus \mathrm{Re}\, T_2$, thus $\alpha \in \sigma(\mathrm{Re}\, T) = \sigma(\mathrm{Re}\, T_1) \cup \sigma(\mathrm{Re}\, T_2)$. Say $\alpha \in \sigma(\mathrm{Re}\, T_k)$; then the application of the preceding paragraph to T_k yields a contradiction. ∎

The following lemma is a special case of a theorem of Putnam [12, Theorem 4], with a considerably simpler proof:

LEMMA 8. *If T satisfies condition (G_1), then* $\mathrm{Re}\, \sigma(T) \subset \sigma(\mathrm{Re}\, T)$.

Proof. We can suppose $\pi(T) = \pi_0(T)$ [2], and therefore $\partial\sigma(T) \subset \pi(T) = \pi_0(T)$.

Suppose $\alpha_0 \in \mathrm{Re}\, \sigma(T)$. Thus, if L is the vertical line $\mathrm{Re}\, \lambda = \alpha_0$, the assumption is that L intersects $\sigma(T)$. Let λ_0 be a point where L exits the spectrum; then $\lambda_0 \in \partial\sigma(T)$ and $\mathrm{Re}\, \lambda_0 = \alpha_0$.

It will suffice to construct a sequence of unit vectors x_n such that $(T - \lambda_0 I)x_n \to 0$ and $(T^* - \lambda_0^* I)x_n \to 0$, for this will imply $\alpha_0 \in \sigma(\mathrm{Re}\, T)$ as in the proof of Lemma 1. To this end, we construct a sequence λ_n of normal eigenvalues of T such that $\lambda_n \to \lambda_0$, as follows.

For $n = 1, 2, 3, \ldots$, let $D_n = \{\lambda : |\lambda - \lambda_0| \leqq 1/n\}$. Since $\lambda_0 \in \partial\sigma(T)$, D_n contains a point μ_n of the resolvent set of T such that $|\mu_n - \lambda_0| < 1/2n$. Clearly dist $(\mu_n, \sigma(T))$ is assumed in D_n; say $\lambda_n \in \sigma(T)$ with dist $(\mu_n, \sigma(T)) = |\mu_n - \lambda_n|$. Thus $\lambda_n \in \sigma(T)$ lies on the circumference of a closed disc (centered at μ_n) whose interior contains no point of $\sigma(T)$; since T satisfies (G_1), it follows that $N(T - \lambda_n I) = N(T^* - \lambda_n^* I)$ [4, Lemma 2]. {The definition of "semibare point" in [4] is unnecessarily restrictive; there is no harm in other points of $\sigma(T)$ lying on the circumference of the disc (in any case, a smaller disc will shake them off).} Obviously $\lambda_n \in \partial\sigma(T)$, so λ_n is an eigenvalue; thus λ_n is a normal eigenvalue. Let x_n be a unit vector with $(T - \lambda_n I)x_n = 0$, $(T^* - \lambda_n^* I)x_n = 0$. Then

$$(T - \lambda_0 I)x_n = (T - \lambda_n I)x_n + (\lambda_n - \lambda_0)x_n = (\lambda_n - \lambda_0)x_n,$$

therefore $\|(T - \lambda_0 I)x_n\| = |\lambda_n - \lambda_0| \leqq 1/n$; thus $(T - \lambda_0 I)x_n \to 0$, and similarly $(T^* - \lambda_0^* I)x_n \to 0$. ∎

Proof of Theorem 3. If $\sigma(T)$ is a spectral set for T, then $\sigma(\mathrm{Re}\, T) \subset \mathrm{Re}\, \sigma(T)$ by Lemma 7; as observed in the proof, T satisfies (G_1), therefore $\mathrm{Re}\, \sigma(T) \subset \sigma(\mathrm{Re}\, T)$ by Lemma 8. ∎

Proof of Theorem 4. Assume T satisfies (G_1) and $\sigma(T)$ is connected. By Lemma 8, Re $\sigma(T) \subset \sigma(\text{Re } T)$. Let $[\alpha, \beta]$ be the smallest interval containing $\sigma(\text{Re } T)$; since T is convexoid [9], [15], it follows from Lemma 5 that $[\alpha, \beta] \subset \text{Re } \sigma(T)$. Thus

$$\sigma(\text{Re } T) \subset [\alpha, \beta] \subset \text{Re } \sigma(T) \subset \sigma(\text{Re } T). \quad \blacksquare$$

If T satisfies the hypotheses of one of the above theorems, then so does $\lambda T + \mu I$ for all complex numbers λ and μ. We have remarked that the operators in Theorems 1–4 are all convexoid. In a sense, this is not an accident: If T is an operator such that $\lambda T + \mu I$ satisfies (*) for all complex λ, μ (equivalently, λT satisfies (*) for all $|\lambda| = 1$), then T is convexoid; the details of the proof are similar to those in Lemmas 5 and 7.

On the other hand, a convexoid operator need not satisfy (*). For example, consider the 5×5 matrix $T = A \oplus B$, where

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and $B = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ is a 3×3 diagonal matrix whose eigenvalues lie off the imaginary axis and are the vertices of a triangle that contains the disc $D_{1/2} = \{\lambda : |\lambda| \leqq 1/2\}$. Since $W(A) = D_{1/2}$ (cf. [7, Problem 166]) and $W(B) = \text{conv}\{\lambda_1, \lambda_2, \lambda_3\} \supset D_{1/2}$, it is easy to see that T is convexoid (cf. [7, p. 113]). One has $\sigma(T) = \sigma(A) \cup \sigma(B) = \{0\} \cup \{\lambda_1, \lambda_2, \lambda_3\}$, and in particular $0 \in \text{Re } \sigma(T)$. On the other hand, $\text{Re } T = \text{Re } A \oplus \text{Re } B$, therefore $\sigma(\text{Re } T) = \sigma(\text{Re } A) \cup \sigma(\text{Re } B) = \{-\frac{1}{2}, \frac{1}{2}\} \cup \text{Re } \sigma(B)$ and so $0 \notin \sigma(\text{Re } T)$. The dimension of the example is optimal, since a convexoid operator on a space of dimension less than 5 is necessarily normal [8, Satz 9, (ii)].

References

1. S. K. Berberian, *Introduction to Hilbert space*, University Texts in the Math. Sciences, Oxford Univ. Press, New York, 1961. MR **25** #1424.

2. ———, *Approximate proper vectors*, Proc. Amer. Math. Soc. **13** (1962), 111–114. MR **24** #A3516.

3. ———, *A note on operators whose spectrum is a spectral set*, Acta Sci. Math. (Szeged) **27** (1966), 201–203. MR **34** #3309.

4. ———, *An extension of Weyl's theorem to a class of not necessarily normal operators*, Michigan Math. J. **16** (1969), 273–279. MR **40** #3335.

5. S. K. Berberian and G. H. Orland, *On the closure of the numerical range of an operator*, Proc. Amer. Math. Soc. **18** (1967), 499–503. MR **35** #3459.

6. J. Dixmier, *Les C*-algèbres et leurs représentations*, Cahiers Scientifiques, fasc. 29, Gauthier-Villars, Paris, 1964. MR **30** #1404.

7. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N. J., 1967. MR **34** #8178.

8. S. Hildebrandt, *Über den numerischen Wertebereich eines Operators*, Math. Ann. **163** (1966), 230–247. MR **34** #613.

9. G. H. Orland, *On a class of operators*, Proc. Amer. Math. Soc. **15** (1964), 75–79. MR **28** #480.

10. C. R. Putnam, *On the spectra of semi-normal operators*, Trans. Amer. Math. Soc. **119** (1965), 509–523. MR **32** #2913.

11. ———, *Commutation properties of Hilbert space operators and related topics*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 36, Springer-Verlag, New York, 1967. MR **36** #707.

12. ———, *The spectra of operators having resolvents of first-order growth*, Trans. Amer. Math. Soc. **133** (1968), 505–510. MR **37** #4651.

13. ———, *An inequality for the area of hyponormal spectra*, Math. Z. (to appear).

14. T. Saitô and T. Yoshino, *On a conjecture of Berberian*, Tôhoku Math. J. (2) **17** (1965), 147–149. MR **32** #4547.

15. J. G. Stampfli, *Hyponormal operators and spectral density*, Trans. Amer. Math. Soc. **117** (1965), 469–476; errata, ibid. **115** (1965), 550. MR **30** #3375; MR **33** #4686.

16. R. Whitley, *The spectral theorem for a normal operator*, Amer. Math. Monthly **75** (1968), 856–861. MR **38** #5040.

17. H. Widom, *On the spectrum of a Toeplitz operator*, Pacific J. Math. **14** (1964), 365–375. MR **29** #476.

18. J. P. Williams, *Minimal spectral sets of compact operators*, Acta Sci. Math. (Szeged) **28** (1967), 93–106. MR **36** #725.

University of Texas,
Austin, Texas 78712